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THE FOURTH DIMENSION.

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In order to describe the position of a point upon a line, we usually proceed as follows. We place a scale, like that of a thermometer, next to the line and make note of the scale-reading which corresponds to the point considered. As in the case of the thermometer this scale reading will be positive or negative according as the point is situated on one side or the other of the zero point of the scale. It thus becomes possible to characterize the position of a point of a line by means of a *single* positive or negative number. This is what is meant by the statement that a line has only *one dimension.*

If we wish to describe the position of a point on a surface, we shall have to employ two numbers. A familiar illustration of this fact is given by the ordinary method of describing the position of a point on the earth's surface by means of its latitude and longitude. These two numbers are called the coordinates of the point. The fact that *two* numbers are required is equivalent to the statement that the surface has *two dimensions.*

But if we take into account the fact that there are mountains and valleys, it becomes clear that a third coordinate becomes necessary which shall express the altitude of a point above, or its depression below sea-level. Obviously three numbers will suffice to determine the position of any point in space. This fact, expressed in another way, leads to the statement that *space has three dimensions.*

Nevertheless, this property of space of having three dimensions is not really a property of space at all, but depends upon our way of looking at it which, as we shall see, is largely due to physiological causes and has no logical foundation whatever. In fact, a number applied to an object of experience always implies that something has been taken as a unit. If we say that the length of a line is expressed by the number five, we can attach no meaning to such an assertion unless we know what unit of measurement has been employed. In the same way, the statement that space has three dimensions is incomplete and meaningless, although our above discussion may be taken to supplement and explain its meaning. The reader may have noticed that we discussed only the method of determining *points* in space, as though space could contain nothing else. To be sure we *may* think of space as an aggregate of points, and analyze all geometric configurations into point-elements; but there is no *logical necessity* for doing so. Let us, for instance, think of space as the assemblage of all of its spheres. This does not change the nature of space in the least; the question is merely whether we prefer to analyse the configurations of geometry into point elements or into sphere elements. As a point-aggregate, space has three dimensions, because it requires three numbers (coordinates) to determine one of its elements (points); As a sphere-aggregate, space has four dimensions, because it requires four numbers (coordinates) to determine one of its elements (spheres). That this latter statement is true can be seen at once, since it takes three numbers (coordinates) to determine the center of a sphere, and a fourth to determine its radius.

A second illustration is of even greater importance. *Space, considered as an aggregate of straight lines, has four dimensions.* To prove this, it suffices to show that four numbers are required to determine a straight line in space. Let us choose any two planes in space, infinite in extent. Two numbers are required to determine a point in one of them, two more to determine a point in the other. The line determined by these two points, therefore, requires four numbers for its identification; *i. e.*, with the straight line as space-element, space has four dimensions.

It may be urged that it is absurd, useless, and artificial to regard space as anything except as an aggregate of points. That it is neither absurd nor useless is conclusively demonstrated by the fact that some very important branches of geometry, finding applications in such practical mattess as steel frame and bridge construction, owe a great part of their recent progress to this point of view. That it seems artificial to analyze the configurations of space into line-elements rather than point-elements, may be regarded as a mere accident of human physiology rather than as a matter of logic. That this is true will perhaps become apparent from the following argument.

Let us imagine a rational being whose body is incapable of motion. His space ideas will therefore be based entirely upon sight impressions, except in so far as he becomes conscious of the muscular efforts which accompany the act of adjusting his eyes to the proper focus for objects at different distances. These muscular sensations, combined with those which arise from the change in the angle of convergence of the two eyes when fixed upon objects at different distances, and the stereoscopic effect, will enable him to arrive at the idea of distance; that of direction is furnished directly by the sense of sight. Let us suppose, however, that his eyes, instead of being, like our own, approximately spherical, are cylindrical in shape. The image of a point upon his retina would be a line. Would it not be natural for him to take the line as his space-element rather than the point? If we now endow him with sense organs in all other respects similar to our own, and also permit him to move about, the notion of a point would also present itself to him, in a secondary way however, and brought to his attention principally by the sense of touch. He would observe that the line image of a point rotates in a certain way when he inclines his head to the right or left; so that, by associating this fact with the idea of a point obtained by the sense of touch, he would come to look upon a point as that geometric configuration which is common to all of the lines which pass through it. He would analyze the point into line-elements, just as we analyze the line into point-elements.

We have indicated the existence of two distinct four-dimensional geometries without leaving the realm of ordinary space. But the question may be asked: is a space conceivable which, considered as an aggregate of *points*, has four dimensions? Probably it is this phase of the question of which most persons think when the subject of the fourth dimension comes under discussion.

That such a space is conceivable in a *logical, purely mathematical sense* is evident from what has been said. Abstractly, *i. e.*, without reference to concrete sense impressions, a point may be defined as any concept which is determined by a set of numbers. We, therefore, obtain at once a four-dimensional point-space in the abstract sense, by choosing the space-element of ordinary space in such a way as to give rise to a four-dimensional geometry, and then substituting the word *point* for the word *space-element.* If it be desired, howover, to obtain a more concrete illustration to show the possibility of the *objective existence* of a four dimensional point-aggregate, we shall be compelled to analyze somewhat more closely the psychological foundations of our space concept. This will lead to some speculations which, while highly imaginative, are not illogical.

It is a familiar notion that those properties of things which we call tone and color do not exist objectively. The vibrations of the air or of the ether, converted into nervous energy and transmitted to the brain, produce upon the mind the peculiar impressions of tone and color. Many philosophers claim that space, likewise, has no objective existence. The spacial relation, according to them, is to be looked upon as something which the mind creates in reacting upon the stimuli of the external world; the mind creates space as it does tone and color. If this is true, space is for us such as we see it; not because it *is* so, but because we *see* it so. If there existed a race of men, or even a single rational being whose space concept involved more than three dimensions, that new kind of space would have the same claim to existence as the old.

Since our argument clearly becomes meaningless on this basis, let us take the point of view that an absolute space exists, and that it has intrinsic properties which are not created by the mind. We shall attempt to show that, even on that assumption, beings somewhat differently organized from ourselves could develop a space concept vastly different from our own. One such case has already been indicated.

The mere fact that we find it difficult to conceive of what the fourth dimension would "look like," no more disproves its existence than the inability of a blind man to conceive of color can be held to show that no such thing exists. It is not hard to imagine a race of beings (flat landers), living in a plane, to whom a third dimension might appear as mysterious as a fourth does to us. Suppose that such a flatlander were to surround an identifiable object of his infinitely thin world by a circle. Nothing can be more obvious to him than the proposition that this object can never reach the outside of the circle without crossing the circumference. But suppose that he sees it disappear, and reappear a moment later outside. To us there is nothing mysterious about this; the object has described a path in space. The flatlander, however, would have to disbelieve his senses, call to his assistance the supernatural, or adopt the hypothesis of a three-dimensional space, of which his own two-dimensional world is but a part. It has been proposed to explain some of the phenomena alleged to have been observed by the spiritualists on the assumption that space has four dimensions. A four dimensional being, for instance, could easily escape from a hermetically closed room. For him, a sphere is no more effective as a barrier than is a circle for us. We see then the possibility of obtaining experimental evidence for the objective existence of a fourth dimension, without wishing to assert that such evidence has already been furnished.

But we may look at the question in still another way. It can hardly be denied that our space concept would be much modified if our sense organs were different from what they are. What, in fact, are the experiences upon which our own ideas of space are based? Obviously the sense of sight is an important factor. It is clear, however, that our geometry would be two dimensional if we were stationary beings, rooted to the ground like a tree, provided with but a single immobile eye in focus for all distances, or if the act of accommodation were not accompanied by muscular sensations. We should be deprived of the notion of distance. The existence of an arm, an organ of touch, etc., enables us to enlarge our idea of space so as to include the notion of distance. Our space concept, then, is an abstraction which unites for us in a consistent manner, a number of qualitatively different sense impressions.

But we do not unite all of our sense-impressions in this way. The notion of color, for instance, does not differ qualitatively from that of direction even as much as does the notion of distance, being a result of pure sight impressions, while that of distance is not. It is quite conceivable that color is no more an intrinsic property of an object than is its distance from us. It is not absurd to imaginine that bodies are actually immersed in a space of four dimensions, but that only three of these present themselves to us as such, the fourth manifesting itself to us as color. We can imagine a being so organized that differences of color, *i. e.*, differences in the rapidity of ether vibrations, would present themselves to him somewhat as distances from a fixed plane do to us. His intuition would, in a sense, be closer to the truth than ours. For, he would see at a glance that the differences which we attribute to color are relations of a quantitative rather than of a qualitative nature, an idea at which *we* can arrive only as a consequence of elaborate theories and complicated experiments.

Without asserting such to be the case, we cannot dismiss offhand the possibility that absolute space, if it exists at all, has more than three dimensions, that these additional dimensions do not however present themselves to us as such, but become sensible in a masked form as color, electric charges, etc. There is no logical reason why a mathematical theory of color or of electricity might not be built upon such a basis.

Unless we assume dogmatically that space is exactly as it appears to us because we create it with all of its properties, so that objectively it does not exist at all, and unless we deny, dogmatically, the possibility that some other race of thinking beings might possess a space concept also subjective, but different from ours, there remains only one possible way of saving the proposition that space cannot have more than three dimensions, and that And that would be the trivial method of defining it in that way.

ON THE NUMERICAL FACTORS OF CERTAIN ARITHMETIC FORMS.*

By R. D. CARMICHAEL, Princeton University.

The methods used by Dickson in an article on the cyclotomic function† may be applied with some change so as to obtain a set of more general propositions than those at which he arrives and of such nature that his theorems are the simplest cases of those thus obtained. The object of this paper is to generalize the theorems of Dickson; and his method of proof will be freely employed.

We shall let

$$Q_n(x)=0$$

be the equation whose roots are the primitive nth roots of unity without repetition. Then $Q_n(x)$ is a polynomial in x with integral coefficients, that of the highest power being unity. The degree of the equation is $\phi(n)$, $\phi(n)$ being Euler's function.

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*Read before the American Mathematical Society, February 29, 1908.

†THE AMERICAN MATHEMATICAL MONTHLY, Vol. XII, pp. 86—89.